This is satisfied if the following conditions hold,

(i) $\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)=0$

(ii) $\int_{\partial \mathcal{R}}\left(-\frac{\partial F}{\partial u_y}dx+\frac{\partial F}{\partial u_x}dy\right)\delta u=0$ (8.40)

The equation (8.40 i) is the *Euler* equation. If u is prescribed on $\partial\mathcal{R}$ i.e. $\delta u=0$ then the equation (8.40 ii) is satisfied otherwise when u is not specified on $\partial\mathcal{R}$, we have

$$-\frac{\partial F}{\partial u_y}\cos\nu+\frac{\partial F}{\partial u_x}\sin\nu=0 \tag{8.41}$$

where ν is the angle which the outward normal to the boundary $\partial\mathcal{R}$ makes with the x axis.
The conditions (8.41) are called *natural* or *suppressible* boundary conditions.

8.3.1 Ritz method

In order to solve a given boundary value problem by the *Ritz* method, we try to write the differential equation as the Euler equation of some variational problem. This will give the appropriate expression for $J[u]$. We now reduce this variational problem to a simple minimizing problem by assuming an approximate function in the form (8.5). Substituting (8.5) in (8.22), we get $J[w]$ as a function of the unknowns $a_1, a_2, \ldots, a_N$. For minimizing $J[w]$, we have

$$\frac{\partial J[w]}{\partial a_j}=\int_a^b\left(\frac{\partial F}{\partial w'}\psi_j'+\frac{\partial F}{\partial w}\psi_j\right)dx=0, j=1, 2, \ldots, N \tag{8.42}$$

which gives N equations in N unknowns. If $\psi_j(x)$ possess continuous second order derivatives, then integrating by parts the first term in the integrand of (8.42) we get

$$\int_a^b \psi_j\left[-\frac{d}{dx}\left(\frac{\partial F}{\partial w'}\right)+\frac{\partial F}{\partial w}\right]dx=0, j=1, 2, \ldots, N \tag{8.43}$$

Equations (8.43) are identical with the Galerkin equations (8.16) for differential equations, which are identical with the Euler equation (8.28). For the differential equation

$$-\frac{d}{dx}(pu')+qu=r(x) \tag{8.44}$$

with boundary conditions (8.23), it can be easily verified that with

$$F=pu'^2+qu^2-2ru$$

the end points x_i and x_{i+1} such that the length of the element (e) is unit may be written as

$$x = x_i + (x_{i+1} - x_i)\xi$$
$$= (1-\xi)x_i + \xi x_{i+1} \tag{8.49}$$

From (8.49) and (8.47), we get

(i) $\xi = \dfrac{x - x_i}{x_{i+1} - x_i} = N_{i+1}(x)$

and

(ii) $1-\xi = \dfrac{x_{i+1} - x}{x_{i+1} - x_i} = N_i(x)$ (8.50)

The transformation (8.49) transforms or maps an element (e) along the x-axis into a *standard interval* [0, 1]. Similarly, if we choose the mid-point of the element (e) as the origin of the ξ-axis then the transformation

$$x = \frac{1}{2}(x_i + x_{i+1}) + \frac{1}{2} l^{(e)}\xi \tag{8.51}$$

maps the subinterval $[x_i, x_{i+1}]$ into a standard interval $[-1, 1]$, where $l^{(e)} = x_{i+1} - x_i$ is the length of the element (e).

The functions ξ and $(1-\xi)$ in (8.50) are ratios of lengths and are called *length, local* or *natural coordinates*. We denote $(1-\xi)$ and ξ by L_i and L_{i+1}, respectively. The coordinates $L_i(x)$ and $L_{i+1}(x)$ are not independent since we have

$$L_i(x) + L_{i+1}(x) = 1 \tag{8.52}$$

The equation (8.49) can also be written as

$$x = L_i(x)x_i + L_{i+1}(x)x_{i+1} \tag{8.53}$$

which shows that the mapping (8.49) is also an interpolation scheme that gives the x coordinate of any point on the element (e) when the corresponding L_i and L_{i+1} coordinates are known. The variation of (L_i, L_{i+1}) inside the element (e) is shown in Figure 8.2(b). Using (8.52) and (8.53), we obtain

$$\begin{bmatrix} L_i \\ L_{i+1} \end{bmatrix} = \begin{bmatrix} x_i & x_{i+1} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ 1 \end{bmatrix} = \frac{1}{(x_{i+1} - x_i)} \begin{bmatrix} -1 & x_{i+1} \\ 1 & -x_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} N_i \\ N_{i+1} \end{bmatrix}$$

Thus we find that in the case of the linear piecewise approximate function the local coordinates are also the shape functions.

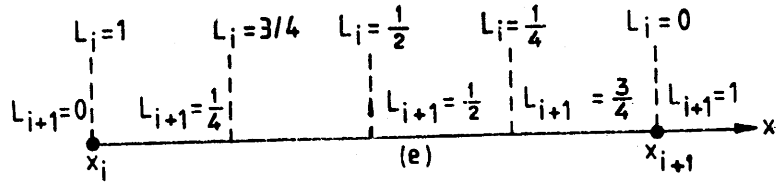

Fig. 8.2(b) Variation of length coordinates within element

The following differentiation and integration results hold over the element (e)

(i) $\dfrac{\partial L_i}{\partial x}=-\dfrac{1}{x_{i+1}-x_i}$, $\dfrac{\partial L_{i+1}}{\partial x}=\dfrac{1}{x_{i+1}-x_i}$

$$\text{(ii)} \int_{x_i}^{x_{i+1}} L_i^r L_{i+1}^t\, dx=\frac{r!\,t!\,(x_{i+1}-x_i)}{(r+t+1)!} \tag{8.54}$$

where r and t are positive integers.

Cubic Hermite polynomial

If the nodal values u_i and u_{i+1} and the first derivative u_i' and u_{i+1}' are used in constructing the piecewise polynomial over the element (e) $(x_i \leqslant x \leqslant x_{i+1})$ then the degree of the interpolating polynomial is three and is called the piecewise *Hermite cubic* polynomial. We have

$$u^{(e)}(x)=N_i(x)u_i+H_i(x)u_i'+N_{i+1}(x)u_{i+1}+H_{i+1}(x)u_{i+1}' \tag{8.55}$$

where

$$N_i(x_i)=1,\quad N_i'(x_i)=0,\quad H_i(x_i)=0\quad H_i'(x_i)=1$$
$$N_{i+1}(x_{i+1})=1,\ N_{i+1}'(x_{i+1})=0,\ H_{i+1}(x_{i+1})=0,\ H_{i+1}'(x_{i+1})=1 \tag{8.56}$$

The shape functions N_i, N_{i+1}, H_i and H_{i+1} may be expressed in terms of L_i and L_{i+1} variables. We obtain

$$\begin{aligned} N_i(x)&=L_i^2(3-2L_i)\\ H_i(x)&=(x_{i+1}-x_i)L_i^2L_{i+1}\\ N_{i+1}(x)&=L_{i+1}^2(3-2L_{i+1})\\ H_{i+1}(x)&=-(x_{i+1}-x_i)L_iL_{i+1}^2 \end{aligned} \tag{8.57}$$

If the piecewise cubic polynomial (8.55) has the second derivative continuous at x_i, $i=1(1)N$, then it is called the *cubic spline* function (Def. 4.2). The continuity of the second derivative at x_i gives the relation

$$\begin{aligned}\frac{1}{h_i}m_{i-1}+\left(\frac{2}{h_i}+\frac{2}{h_{i+1}}\right)m_i+\frac{1}{h_{i+1}}m_{i+1}\\ =-3\frac{u_{i-1}-u_i}{h_i^2}+3\frac{u_{i+1}-u_i}{h_{i+1}^2}\end{aligned} \tag{8.58}$$

where $\mathbf{N}^{(e)} = [N_1 N_2 N_3], \quad \boldsymbol{\phi}^{(e)} = [u_1\ u_2\ u_3]^T$

$$N_1(x, y) = \frac{1}{2\Delta^{(e)}}(a_1 + b_1 x + c_1 y)$$

$$N_2(x, y) = \frac{1}{2\Delta^{(e)}}(a_2 + b_2 x + c_2 y)$$

$$N_3(x, y) = \frac{1}{2\Delta^{(e)}}(a_3 + b_3 x + c_3 y)$$

$$\begin{aligned} a_1 &= x_2 y_3 - x_3 y_2, & b_1 &= y_2 - y_3, & c_1 &= x_3 - x_2 \\ a_2 &= x_3 y_1 - x_1 y_3, & b_2 &= y_3 - y_1, & c_2 &= x_1 - x_3 \\ a_3 &= x_1 y_2 - x_2 y_1, & b_3 &= y_1 - y_2, & c_3 &= x_2 - x_1 \end{aligned} \tag{8.64}$$

The functions $N_i(x, y)$, $i = 1, 2, 3$ are called *shape functions* and defined in the element (e). It is easily verified from (8.64) that

$$N_i(x_j, y_j) = \delta_{ij} \tag{8.65}$$

where δ_{ij} is a kronecker delta, $\delta_{ij} = 1$, $i = j$, $\delta_{ij} = 0$, $i \neq j$.

A point $P(x, y) \in (e)$ can be associated with the *area coordinates* (L_1, L_2, L_3) defined by

$$L_1 = \frac{\text{area } P23}{\text{area } 123}, \quad L_2 = \frac{\text{area } P31}{\text{area } 123}, \quad L_3 = \frac{\text{area } P12}{\text{area } 123} \tag{8.66}$$

as shown in Figure 8.3(b). The area coordinate system is also called *local* or *natural coordinate* system. From (8.64) and (8.66) we find that the area coordinates satisfy the following relations

(i) $L_1 + L_2 + L_3 = 1$

$$L_i > 0 \quad i = 1, 2, 3$$

(ii) $L_1 = N_1(x, y)$, $L_2 = N_2(x, y)$, $L_3 = N_3(x, y)$

(iii) $$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \frac{1}{2\Delta^{(e)}} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

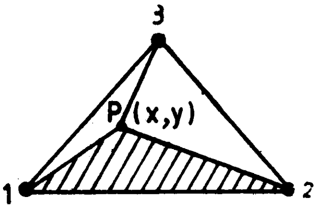

Fig. 8.3(b) Representation of local coordinates

or

$$\text{(iv)} \quad \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \tag{8.67}$$

Although three coordinates L_1, L_2 and L_3 are used to define a point P, only two are independent since they must satisfy (8.67 i). The variation of (L_1, L_2, L_3) inside an element (e) is shown in Figure 8.3(c). The equations of the sides 2-3, 3-1 and 1-2 respectively are given by

$$L_1=0, \quad L_2=0, \quad L_3=0 \tag{8.68}$$

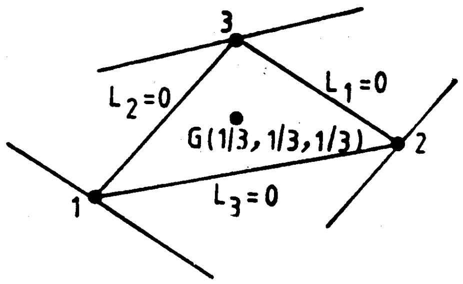

Fig. 8.3(c) Variation of local coordinates

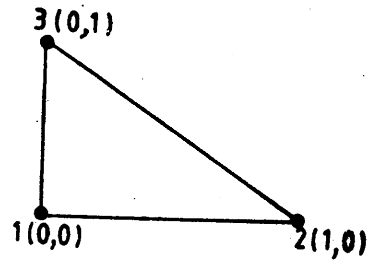

Fig. 8.3(d) Standard triangle

The area coordinates of the nodes 1, 2 and 3 of the element (e) are (1, 0, 0), (0, 1, 0) and (0, 0, 1), respectively. Further, the relation (8.67 iv) may be written as

$$x=(1-L_2-L_3)x_1+x_2L_2+x_3L_3$$

$$y=(1-L_2-L_3)y_1+y_2L_2+y_3L_3 \tag{8.69}$$

which transforms or maps the element (e) in x, y coordinates into a *standard triangle* as shown in Figure 8.3(d).

The differentiation of $N_i(L_1, L_2, L_3)$ with respect to x and y may be written as

$$\frac{\partial N_i}{\partial x}=\frac{\partial N_i}{\partial L_1}\frac{\partial L_1}{\partial x}+\frac{\partial N_i}{\partial L_2}\frac{\partial L_2}{\partial x}+\frac{\partial N_i}{\partial L_3}\frac{\partial L_3}{\partial x}$$

$$\frac{\partial N_i}{\partial y}=\frac{\partial N_i}{\partial L_1}\frac{\partial L_1}{\partial y}+\frac{\partial N_i}{\partial L_2}\frac{\partial L_2}{\partial y}+\frac{\partial N_i}{\partial L_3}\frac{\partial L_3}{\partial y} \tag{8.70}$$

where

$$\frac{\partial L_i}{\partial x}=\frac{b_i}{2\Delta^{(e)}}, \quad \frac{\partial L_i}{\partial y}=\frac{c_i}{2\Delta^{(e)}}, \quad i=1, 2, 3 \tag{8.71}$$

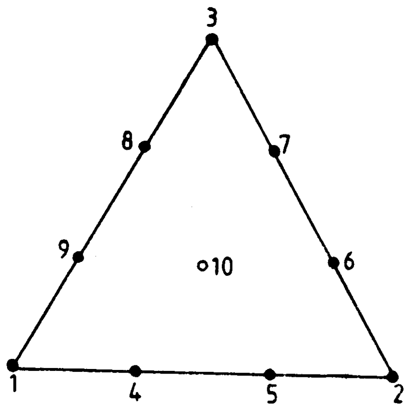

Fig. 8.3(f) Cubic element with ten nodes

The shape functions N_i may be assumed in the form

$$N_i = a_1^{(i)}L_1 + a_2^{(i)}L_2 + a_3^{(i)}L_3 + a_4^{(i)}L_1L_2 + a_5^{(i)}L_2L_3 + a_6^{(i)}L_1L_3 + a_7^{(i)}L_1^2L_2$$
$$+ a_8^{(i)}L_2^2L_3 + a_9^{(i)}L_3^2L_1 + a_{10}^{(i)}L_1L_2L_3 \quad (8.82)$$

Using the interpolating conditions $u^{(e)}(x_i, y_i) = u_i$, $i = 1(1)10$ and (8.75), we obtain

$$\begin{aligned} &N_i = \tfrac{1}{2}L_i(3L_i - 1)(3L_i - 2), \quad i = 1, 2, 3 \\ &N_4 = \tfrac{9}{2}L_1L_2(3L_1 - 1), \qquad N_6 = \tfrac{9}{2}L_2L_3(3L_2 - 1) \\ &N_5 = \tfrac{9}{2}L_1L_2(3L_2 - 1), \qquad N_9 = \tfrac{9}{2}L_1L_3(3L_1 - 1) \\ &N_7 = \tfrac{9}{2}L_2L_3(3L_3 - 1), \qquad N_8 = \tfrac{9}{2}L_1L_3(3L_3 - 1) \\ &N_{10} = 27L_1L_2L_3 \end{aligned} \quad (8.83)$$

We may use the relation

$$u_{10} + \tfrac{1}{6}(u_1 + u_2 + u_3) - \tfrac{1}{4}(u_4 + u_5 + u_6 + u_7 + u_8 + u_9) = 0 \quad (8.84)$$

to eliminate u_{10} from (8.81) and obtain the piecewise cubic Lagrange interpolation polynomial dependent on nine nodal values u_i, $i = 1(1)9$.

Cubic Hermite polynomial

Here we construct a cubic polynomial by specifying u, $\partial u/\partial x$ and $\partial u/\partial y$ at the nodes 1, 2 and 3 of the element (e) and u at an interior node 4, as shown in Figure 8.3(g). We use for (8.73), $p = 3$, an alternative expression in the form

$$u^{(e)}(x, y) = \sum_{i=1}^{3}\left[N_i(x, y)u_i + H_i(x, y)\left(\frac{\partial u}{\partial x}\right)_i + K_i(x, y)\left(\frac{\partial u}{\partial y}\right)_i\right] + N_4u_4 \quad (8.85)$$
$$= [N_1H_1K_1N_2H_2K_2N_3H_3K_3N_4]\phi^{(e)}$$

where

$$\phi^{(e)} = [u_1\ u_{x_1}\ u_{y_1}\ u_2\ u_{x_2}\ u_{y_2}\ u_3\ u_{x_3}\ u_{y_3}\ u_4]^T$$

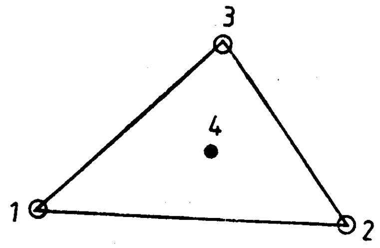

Fig. 8.3(g) Cubic Hermite element
○ u, u_x, u_y prescribed
● u prescribed

The functions N_i, H_i and K_i are the shape functions to be determined by satisfying the interpolation conditions. We obtain

$$\begin{aligned}
N_1 &= L_1^2(3-2L_1) - 7L_1L_2L_3\\
H_1 &= L_1^2(c_3L_2 - c_2L_3) + (c_2 - c_3)L_1L_2L_3\\
K_1 &= L_1^2(b_2L_3 - b_3L_2) + (b_3 - b_2)L_1L_2L_3\\
N_2 &= L_2^2(3-2L_2) - 7L_1L_2L_3\\
H_2 &= L_2^2(c_1L_3 - c_3L_1) + (c_3 - c_1)L_1L_2L_3\\
K_2 &= L_2^2(b_3L_1 - b_1L_3) + (b_1 - b_3)L_1L_2L_3\\
N_3 &= L_3^2(3-2L_3) - 7L_1L_2L_3\\
H_3 &= L_3^2(c_2L_1 - c_1L_2) + (c_1 - c_2)L_1L_2L_3\\
K_3 &= L_3^2(b_1L_2 - b_2L_1) + (b_2 - b_1)L_1L_2L_3\\
N_4 &= 27L_1L_2L_3
\end{aligned} \tag{8.86}$$

where b_i's and c_i's are given in (8.64).

8.4.3 Rectangular element

The rectangular network has already been used to develop difference methods for the solution of partial differential equations. We discretize the domain $\mathcal{R}$ (see Figure 7.1) by drawing lines parallel to the x and y axes. We take an arbitrary rectangular element (e) as shown in Figure 8.4(a) and denote the value of the function $u(x, y)$ at the node i by u_i.

Linear Lagrange polynomial

Here the piecewise polynomial chosen is of the form

$$u^{(e)}(x, y) = a_1 + a_2x + a_3y + a_4xy \tag{8.87}$$

where a_i's are defined for each element. Using the interpolating conditions $u_i^{(e)} = u_i$, $i = 1(1)4$ and solving the resulting linear equations, we may determine

Quadratic Lagrange polynomial

The choice of the nodes on the rectangular element (e) is shown in Figure 8.4(c). The approximate piecewise function can be written as

$$u^{(e)}(x, y) = \sum_{i=1}^{8} N_i(x, y)\, u_i \tag{8.94}$$

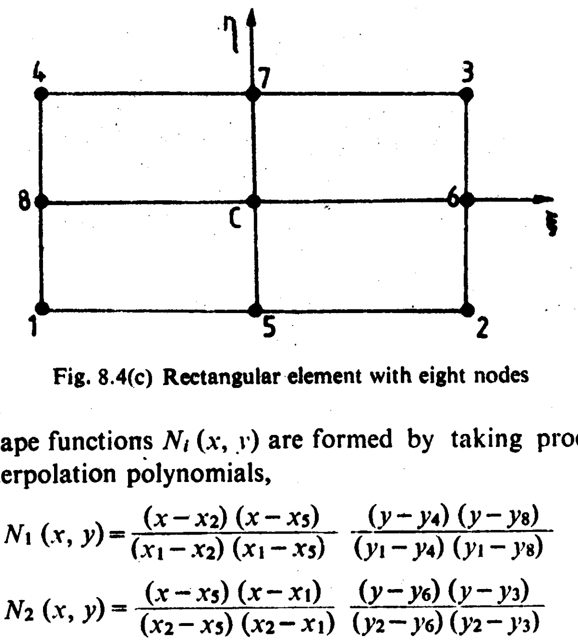

Fig. 8.4(c) Rectangular element with eight nodes

where the shape functions $N_i(x, y)$ are formed by taking products of the Lagrange interpolation polynomials,

$$N_1(x, y) = \frac{(x-x_2)(x-x_5)}{(x_1-x_2)(x_1-x_5)} \frac{(y-y_4)(y-y_8)}{(y_1-y_4)(y_1-y_8)}$$

$$N_2(x, y) = \frac{(x-x_5)(x-x_1)}{(x_2-x_5)(x_2-x_1)} \frac{(y-y_6)(y-y_3)}{(y_2-y_6)(y_2-y_3)}$$

$$N_5(x, y) = \frac{(x-x_1)(x-x_2)}{(x_5-x_1)(x_5-x_2)} \frac{(y-y_7)}{(y_5-y_7)}, \text{ etc.}$$

The shape functions $N_i(x, y)$ in ξ, η coordinate system become at corner nodes $i = 1, 2, 3, 4$

$$N_i(\xi, \eta) = \tfrac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)$$

on mid-side nodes $\quad i = 5, 7$

$$N_i(\xi, \eta) = \tfrac{1}{2}(1-\xi^2)(1+\eta\eta_i)$$

on mid-side nodes $\quad i = 6, 8$

$$N_i(\xi, \eta) = \tfrac{1}{2}(1+\xi\xi_i)(1-\eta^2) \tag{8.95}$$

Cubic Lagrange polynomial

The distribution of nodes on the rectangular element (e) is shown in Figure 8.4(d). The piecewise cubic polynomial is given by

$$u^{(e)}(x, y) = \sum_{i=1}^{12} N_i(x, y)\, u_i \tag{8.96}$$

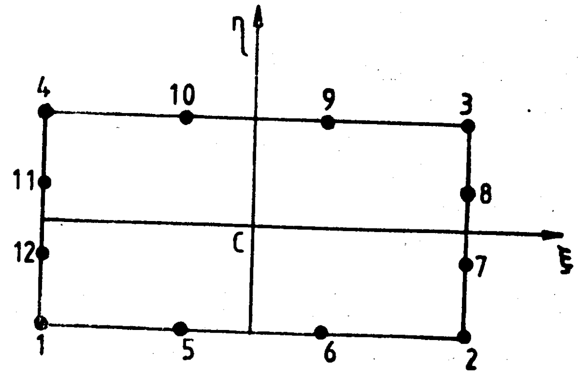

Fig 8.4(d) Rectangular element with twelve nodes

where the shape functions N_i in ξ, η coordinates become at corner nodes $i=1, 2, 3, 4$

$$N_i(\xi, \eta)=\frac{1}{32}(1+\xi\xi_i)(1+\eta\eta_i)[9(\xi^2+\eta^2)-10]$$

on mid-side nodes $i=7, 8, 11, 12$

$$N_i(\xi, \eta)=\frac{9}{32}(1+\xi\xi_i)(1-\eta^2)(1+9\eta\eta_i)$$

and on mid-side nodes $i=5, 6, 9, 10$

$$N_i(\xi, \eta)=\frac{9}{32}(1+\eta\eta_i)(1-\xi^2)(1+9\xi\xi_i) \tag{8.97}$$

Cubic Hermite polynomial

The function and its first order partial derivatives $u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are specified at the four corner nodes of the rectangular element (e) as shown in Figure 8.4(e). Satisfying the interpolation conditions, we get the cubic Hermite polynomial

$$u^{(e)}(x, y)=\sum_{i=1}^{4}\left[N_i(x, y)\,u_i+H_i(x, y)\left(\frac{\partial u}{\partial x}\right)_i+K_i(x, y)\left(\frac{\partial u}{\partial y}\right)_i\right] \tag{8.98}$$

where the shape functions N_i, H_i and K_i in ξ, η coordinates are given by

$$N_1(\xi, \eta)=\frac{1}{8}(\eta-1)(\xi-1)\left[\frac{1}{2}(\eta-1)(\xi-1)-\frac{1}{2}(\eta+1)(\xi+1)\right.$$
$$\left.-(\eta+1)(\eta-1)-(\xi+1)(\xi-1)\right]$$

$$H_1(\xi, \eta)=\frac{a}{8}(\eta-1)(\xi-1)^2(\xi+1)$$

$$K_1(\xi, \eta)=-\frac{b}{8}(\eta-1)^2(\xi-1)(\eta+1)$$

8.4.5 Tetrahedron element

We discretize the three dimensional region $\mathscr{R}$ using tetrahedron elements. We consider an arbitrary tetrahedron element (e) with four nodes labelled 1, 2, 3 and 4 as shown in Figure 8.6(a). The nodes 1, 2 and 3 are labelled in a counter clockwise sequence when veiwed from the node 4. The value of the function $u(x, y, z)$ at the node i is represented by u_i.

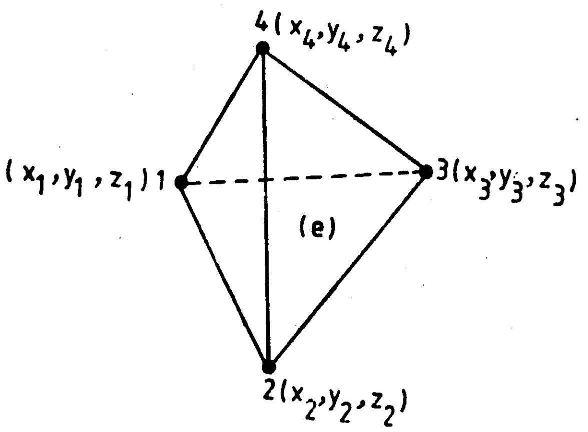

Fig. 8.6(a) Tetrahedron element

Linear Lagrange polynomial

The linear piecewise polynomial in each element has the form

$$u^{(e)} = a + bx + cy + dz = [1\ x\ y\ z]\,\mathbf{a} \tag{8.100}$$

where $\mathbf{a} = [a\ b\ c\ d]^T$ is to be determined using the interpolation conditions

$$u^{(e)}(x_i, y_i, z_i) = u_i, \qquad i = 1(1)4$$

We have

$$\begin{aligned} u_1 &= a + bx_1 + cy_1 + dz_1 \\ u_2 &= a + bx_2 + cy_2 + dz_2 \\ u_3 &= a + bx_3 + cy_3 + dz_3 \\ u_4 &= a + bx_4 + cy_4 + dz_3 \end{aligned} \tag{8.101}$$

Solving for a, b, c and d, we obtain

$$\begin{aligned} a &= \frac{1}{6V^{(e)}}(a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4) \\ b &= \frac{1}{6V^{(e)}}(b_1u_1 + b_2u_2 + b_3u_3 + b_4u_4) \\ c &= \frac{1}{6V^{(e)}}(c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4) \\ d &= \frac{1}{6V^{(e)}}(d_1u_1 + d_2u_2 + d_3u_3 + d_4u_4) \end{aligned} \tag{8.102}$$